ALGEBRAIC CURVES : SOLUTIONS SHEET 5

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let $n \geq 1$ and $I, J \subseteq k[X_0, \ldots, X_n]$ be ideals. For $d \geq 0$ we denote by $k[X_0, \ldots, X_n]_d$ the subspace of forms of degree d and $I_d = I \cap k[X_0, \ldots, X_n]_d$ (resp. $J_d = J \cap k[X_0, \ldots, X_n]_d$). Show that:

(1) If I, J are homogeneous, then $I + J$, IJ and $rad(I)$ are homogeneous.
(2) If I is homogeneous, I is prime if, and only if, for all *homogeneous* $f, g \in k[X_0, \ldots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$.
(3) I is homogeneous if, and only if, $I = \bigoplus_{d \geq 0} I_d$ (the right-hand side being a direct sum of abelian groups). Give an example of how this fails for non-homogeneous ideals.
(4) If I is homogeneous, then there is a well-defined notion of forms of degree d in $\Gamma = k[X_0, \ldots, X_n]/I$ and the corresponding spaces Γ_d, $d \geq 0$ are finite-dimensional over k.

Solution 1.

(1) Assume I, J homogenous. Fix homogenous generators $f_1, \ldots, f_n$ of I and $g_1, \ldots, g_n$ of J. Then

$$I + J = \langle f_i, g_j \mid i, j \rangle$$
$$IJ = \langle f_i g_j \mid i, j \rangle$$

are also homogenous, as products of homogenous polynomials are also homogenous.

Finally, to prove that $\mathrm{Rad}(I)$ is homogeneous, recall that a polynomial is contained in a homogeneous ideal I if and only if all of its homogeneous parts are contained in I (Lemma 3.1). Let $F = F_0 + \cdots + F_d$ be an element of $\mathrm{Rad}(I)$ where F_i is the homogeneous part of degree i. Suppose by contradiction that there exists i with $F_i \notin \mathrm{Rad}(I)$. If d' is the maximal such i, then

$$F_0 + \cdots + F_{d'} = \underbrace{F}_{\in \mathrm{Rad}(I)} - \underbrace{F_d}_{\in \mathrm{Rad}(I)} - \cdots - \underbrace{F_{d'+1}}_{\in \mathrm{Rad}(I)} \in \mathrm{Rad}(I).$$

Hence we may assume that $d' = d$, i.e. $F_d \notin \mathrm{Rad}(I)$. Let now $N > 0$ be large enough so that $F^N \in I$. As I is homogeneous, it contains all the homogeneous parts of F^N (by Lemma 3.1). The homogeneous part of degree Nd of F^N is F_d^N, so $F_d^N \in I$. In particular, we have $F_d \in \mathrm{Rad}(I)$, contradiction. Therefore, all the F_i have to be contained in $\mathrm{Rad}(I)$ to begin with, i.e. $\mathrm{Rad}(I)$ is homogeneous.

(2) Clearly, I prime implies for all homogeneous $f, g \in k[X_0, \dots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$.

Now suppose that I is an homogenous ideal, such that for all homogeneous $f, g \in k[X_0, \dots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$. Let $f, g \in k[X_0, \dots, X_n]$ be such that $fg \in I$. Take their decomposition into homogenous components $f = \sum_{i=0}^{d} f_i$ and $g = \sum_{i=0}^{e} g_i$, where f_i, g_i are homogenous of degree i. Assume by contradiction that $f, g \notin I$. Then by Lemma 3.1 there exist d', e' such that $f_{d'}, g_{e'} \notin I$, and as in the proof that $\operatorname{Rad}(I)$ is homogeneous we may suppose that in fact $d' = d$ and $e' = e$. As $fg \in I$, the homogeneous part of degree de of fg, which is $f_d g_e$, is contained in I (again Lemma 3.1). But then by hypothesis either $f_d \in I$ or $g_e \in I$, contradiction. Hence we must have $f \in I$ or $g \in I$.

(3) This is essentially a rephrasement of Lemma 3.1, i.e. that a polynomial is contained in a homogeneous ideal, if and only if all its homogeneous parts are contained in it.

By the universal property of the direct sum, the inclusions $I_d \subseteq I$ give rise to a morphism of abelian groups

$$\Phi\colon \bigoplus_{d \geq 0} I_d \to I$$
$$(f_d)_{d \geq 0} \mapsto \sum_{d \geq 0} f_d.$$

We have to show that if I is homogeneous, then Φ is an isomorphism. It is straightforward to see that Φ is injective, so let us show surjectivity. If $f \in I$ is arbitrary, and $f = \sum_{d \geq 0} f_d$ is its decomposition into homogeneous parts, then by Lemma 3.1 we have $f_d \in I$ for all d. It immediately follows that $f_d \in I_d$ for all d, and so $f = \Phi((f_d)_{d \geq 0})$. Hence Φ is an isomorphism.

The statement fails for non homogeneous ideals: take $J = \langle y - x^2 \rangle \subseteq k[x, y]$. Then y and x^2 are homogeneous components of an element of J but they are not in J. In fact, it is straightforward to see that $J_d = 0$ for all d, so Φ is not surjective.

(4) We define Γ_d to be the vector subspace of elements of Γ which have a representative in $k[X_1, \dots, X_n]_d$, i.e.

$$\Gamma_d := (k[X_1, \dots, X_n]_d + I) \big/ I.$$

Again, by the universal property of the direct sum we obtain an injective homomorphism of abelian groups

$$\Psi\colon \bigoplus_{d \geq 0} \Gamma_d \to \Gamma$$
$$(f_d + I)_{d \geq 0} \mapsto \sum_{d \geq 0} f_d + I.$$

As in point (3), we want to show that Ψ is surjective, and thus an isomorphism. So let $f + I \in \Gamma$ be arbitrary. As usual, we decompose f into homogeneous parts and write $f = \sum_{d \geq 0} f_d$. Then for all d we have $f_d + I \in \Gamma_d$, and clearly $\Psi((f_d + I)_{d \geq 0}) = f + I$. So Ψ is an isomorphism.

Hence, if we call the elements of Γ_d the forms of degree d in Γ, we can decompose every element of Γ into a sum of forms. Furthermore, it is straightforward to see that if $F \in \Gamma_d$ and $G \in \Gamma_e$, then $FG \in \Gamma_{d+e}$. Hence this decomposition enjoys analogous properties to the decomposition of a polynomial into homogeneous parts. Also, we may define the degree $\deg_\Gamma(f + I)$ of an element $f + I \in \Gamma$ as the maximal d such that $f_d + I \neq 0$ (i.e. $f_d \notin I$), with the convention $\deg_\Gamma 0 = -\infty$, and this then has similar properties as the degree function for polynomials.

Finally, to see that Γ_d is finite dimensional over k, note that by one of the isomorphism theorems of modules we have

$$\Gamma_d = (k[X_1, \dots, X_n]_d + I) \big/ I \cong (k[X_1, \dots, X_n]_d) \big/ I_d,$$

so Γ_d is a quotient of the finite dimensional k-vector space $k[X_1, \dots, X_n]_d$.

Remark. There is a name for rings with a decomposition like $k[X_1, \dots, X_n]$ and Γ: a ring R is called *graded*, if for all $d \geq 0$ there exist additive subgroups $R_d \subseteq R$ such that the natural map $\bigoplus_{d \geq 0} R_d \to R$ is an isomorphism, and such that $R_d R_e \subseteq R_{d+e}$ for all $d, e \geq 0$. The elements of R_d are said to be *homogeneous of degree d*. What we showed in point (4) is that the quotient R/I of a graded ring R by a homogeneous ideal I has a natural grading, such that the quotient map $R \to R/I$ respects the grading.

Exercise 2.
Let $R = k[X, Y, Z]$ and $F \in R$ be an irreducible form of degree $n \geq 1$. Consider $V = V(F) \subseteq \mathbb{P}^2_k$ and $\Gamma = R/(F)$. For $d \geq 0$, we denote by Γ_d the subspace of forms of degree d in Γ (see previous exercise).

(1) Construct an exact sequence $0 \to R \stackrel{\times F}{\to} R \to \Gamma \to 0$, where $\times F$ denotes multiplication by F in R.
(2) Show that, for $d > n$:

$$dim_k(\Gamma_d) = dn - \frac{n(n-3)}{2}$$

Solution 2.

(1) To show that $0 \to R \stackrel{\times F}{\to} R \to \Gamma \to 0$ is exact, we can say that :
- $R \to R$, $f \mapsto f \cdot F$ defines a group morphism, injective since $F \neq 0$ and R is a domain. The image is (F).
- (F) is the kernel of the quotient map $R \to \Gamma$. Quotient maps are always surjective.

(2) Notice that $\times F$ sends R_i to R_{n+i}, and the quotient map $R \to \Gamma$ sends R_i to Γ_i. We hence obtain a sequence

$$0 \to R_{d-n} \xrightarrow{\times F} R_d \to \Gamma_d \to 0$$

and it is straightforward to check that this is still exact. Hence

$$\dim_k(\Gamma_d) = \dim_k(R_d) - \dim_k(R_{d-n})$$

To conclude, note that the dimension of forms of degree d in $k[X_0, \dots, X_N]$ is given by $\binom{d+N-1}{N-1}$.

Indeed, a choice of an element of the basis is given by choosing the position of $N-1$ bars separating d stars. For example, " $**|*|*$ " would represent the $4-$form x^2yz.

Now $\binom{d+2}{2} - \binom{d-n+2}{2}$ gives the desired expression.

Exercise 3. Let $V = V(Y - X^2, Z - X^3) \subseteq \mathbb{A}^3_k$. Show that:

(1) $I(V) = (Y - X^2, Z - X^3)$.
(2) $ZW - XY \in I(V)^* \subseteq k[X, Y, Z, W]$, but $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$.

In particular, this shows that, for $F_1, \dots F_r \in k[X_1, \dots, X_n]$, the following inclusion can be strict: $(F_1^*, \dots, F_r^*) \subseteq (F_1, \dots, F_r)^*$.

Solution 3.

(1) Set $I = (Y - X^2, Z - X^3)$. Since $k[X, Y, Z]/I \cong k[X]$ is reduced, I is radical, hence $I = I(V)$.
(2) Note that $Z - XY = Z - X^3 - X(Y - X^2) \in I$, so $ZW - XY = (Z - XY)^* \in I^*$.

However, $(Y - X^2)^* = WY - X^2$, $(Z - X^3)^* = W^2Z - X^3$. Suppose by contradiction that there exist $F, G \in k[X, Y, Z, W]$ such that

$$ZW - XY = F \cdot (WY - X^2) + G \cdot (W^2Z - X^3).$$

Taking degree 2 parts of both sides we obtain $ZW - XY = F_0 \cdot (WY - X^2)$ which is impossible. Hence $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$

Exercise 4. Let $n \geq 1$ and $T : \mathbb{A}^{n+1}_k \to \mathbb{A}^{n+1}_k$ be a linear change of coordinates (i.e. a linear automorphism of k^{n+1}). As it preserves lines through the origin it induces $T : \mathbb{P}^n_k \to \mathbb{P}^n_k$, what we call a *projective change of coordinates*.

(1) Show, that one can send any $n+1$ points in $\mathbb{P}^n$ not lying on a hyperplane to any other $n+1$ points not lying on a hyperplane via a linear change of coordinates.
(2) Formulate and prove a similar statement for hyperplanes instead of points.

Solution 4. We denote the natural quotient map by $[\bullet]\colon k^{n+1}\setminus\{0\}\to\mathbb{P}^n$. For a vector subspace $V\subseteq k^{n+1}$, we denote by $[V]\coloneqq[V\setminus\{0\}]$ its image under $[\bullet]$. Note that every hyperplane of $\mathbb{P}^n$ is of the form $[V]$ for a n-dimensional subspace $[V]$. Indeed, if $h\in k[X_1,\dots,X_{n+1}]$ is linear, then it defines a linear map $h\colon k^{n+1}\to k$, and $V_p(h)=[\ker(h)]$.

(1) Consider $n+1$ points $P_1,\dots,P_{n+1}\in\mathbb{P}^n$ and choose preimages $P_i=[p_i]$. As $P_1,\dots,P_{n+1}\in[\operatorname{span}_k\{p_1,\dots,p_{n+1}\}]$, we must have $\operatorname{span}_k\{p_1,\dots,p_{n+1}\}=k^{n+1}$ (otherwise the points would lie in a hyperplane). That is, $p_1,\dots,p_{n+1}$ is a basis of k^{n+1}.

For another set of points $Q_1,\dots,Q_{n+1}\in\mathbb{P}^n$ with preimages $Q_i=[q_i]$, we also obtain that $q_1,\dots,q_{n+1}$ is a basis of k^{n+1}. Hence if T is the linear map induced by the change of basis $T(p_i)=q_i$, we obtain a projective change of coordinates $T\colon\mathbb{P}^n_k\to\mathbb{P}^n_k$ sending P_i to Q_i.

(2) Consider $n+1$ hyperplanes $H_1,\dots,H_{n+1}\subseteq\mathbb{P}^n$. Let $h_1,\dots,h_{n+1}\in k[X_1,\dots,X_{n+1}]$ be linear forms such that $H_i=V(h_i)$ for all i. Notice that any two linear forms having the same vanishing locus are scalar multiples of each other. Hence if we write

$$h_i=p_{i1}X_1+\cdots+p_{i,n+1}X_{n+1},$$

then the point $P_i=[p_{i1}:\ldots:p_{i,n+1}]$ doesn't depend on the choice of h_i, but only on H_i. To apply point (1), we translate what it means for $H_1,\dots,H_{n+1}$ if $P_1,\dots,P_{n+1}$ don't lie on a common hyperplane: in that case, we saw that $p_1,\dots,p_{n+1}$ form a basis of k^{n+1}, i.e. the $(n+1)\times(n+1)$-matrix with rows $p_1,\dots,p_{n+1}$ is invertible. Hence it has trivial kernel, which is equivalent to say that

$$\{0\}=\bigcap_{i=1}^{n+1}\ker(h_i),$$

which in turn is equivalent to

$$\emptyset=\bigcap_{i=1}^{n+1}H_i.$$

Conversely, if the intersection of the H_i's is empty, this translates back to the kernel of the matrix with rows $p_1,\dots,p_{n+1}$ having trivial kernel, so the rows have to be linearily independent, which means that $P_1,\dots,P_{n+1}$ don't lie on a common hyperplane.

This leads us to formulating the following statement: any $n+1$ hyperplanes $H_1,\dots,H_{n+1}\subseteq\mathbb{P}^n$ with empty intersection can be sent to any other $n+1$ hyperplanes $G_1,\dots,G_{n+1}\subseteq\mathbb{P}^n$ having empty intersection by a projective change of coordinates.

The key to translating this back to point (1) is the following: if h_i are linear forms such that $H_i=V(h_i)$for all i, and if $T\colon\mathbb{A}^{n+1}\to\mathbb{A}^{n+1}$ is any linear change of coordinates, then $T(H_i) = V(h_i \circ T^{-1})$. Note that if $p_i \in \mathbb{A}^{n+1}$ is the column vector containing the coefficients of h_i, then $h_i \circ T^{-1}$ is represented by $(T^{-1})^\intercal \cdot p_i$. Therefore, if $P_i = [p_i] \in \mathbb{P}^n$ is the point representing H_i and $Q_i \in \mathbb{P}^n$ is the point representing G_i, then by point (1) there exists $\widetilde{T}$ such that $\widetilde{T}(P_i) = Q_i$ for all i, and then $T := (\widetilde{T}^\intercal)^{-1}$ will satisfy $T(H_i) = G_i$ for all i.

Remark. The key concept behind the above argument is the following: denote by $(\mathbb{P}^n)^*$ the set of hyperplanes in $\mathbb{P}^n$. Then there is a bijection $(\mathbb{P}^n)^* \cong \mathbb{P}^n$, defined by

$$\begin{aligned} (\mathbb{P}^n)^* &\to \mathbb{P}^n \\ V(p_1X_1 + \cdots + p_{n+1}X_{n+1}) &\mapsto [p_1 : \ldots : p_{n+1}]. \end{aligned}$$

If T is a projective change of coordinates, then T induces a map $(\mathbb{P}^n)^* \to (\mathbb{P}^n)^*$ by the rule $H \mapsto T(H)$. Under the above identification, we saw in the proof that this map induced by T is in fact $(T^{-1})^\intercal$.

In general, this duality between $(\mathbb{P}^n)^*$ and $\mathbb{P}^n$ (in fact we have a canonical isomorphism $\mathbb{P}^n \to (\mathbb{P}^n)^{**}$ whereby a point is mapped to the set of hyperplanes containing it) can be used to transform statements about points in $\mathbb{P}^n$ to statements about hyperplanes and vice versa.

Exercise 5. Show that any two distinct lines in $\mathbb{P}^2_k$ intersect in one point.

Solution 5. Let L and L' be two distinct lines in $\mathbb{P}^2$. Then there are two-dimensional vector subspaces $V, V' \subseteq \mathbb{A}^3$ such that $L = [V]$ resp. $L = [V']$ (see the beginning of Solution 4 for the notation). It is straightforward to check that

$$L \cap L' = [V] \cap [V'] = [V \cap V'].$$

As L and L' are distinct, V and V' are distinct, and thus we must have $V + V' = \mathbb{A}^3$. Hence by the dimension formula we have

$$\dim_k(V \cap V') = \dim_k V + \dim_k V' - \dim_k(V + V') = 2 + 2 - 3 = 1.$$

Hence $V \cap V'$ is a line through the origin, so that $[V \cap V'] = \{P\}$ is a singleton. That is, P is the unique point of intersection of L and L'.

Exercise 6.
Let $m, n \geq 1$ and $N = (n+1)(m+1) - 1 = mn + m + n$. We consider $\mathbb{P}^n_k$ with projective coordinates $X_0, \ldots, X_n$, $\mathbb{P}^m_k$ with projective coordinates $Y_0, \ldots, Y_m$ and $\mathbb{P}^N_k$ with projective coordinates $T_{00}, T_{01}, \ldots, T_{0m}, T_{10}, \ldots, T_{nm}$. We also denote the affine coverings of $\mathbb{P}^n_k$, $\mathbb{P}^m_k$, $\mathbb{P}^N_k$ associated to these coordinates as follows: $U_i = \{X_i \neq 0\}$, $V_j = \{Y_j \neq 0\}$ and $W_{ij} = \{T_{ij} \neq 0\}$.
Define the *Segre embedding* $S : \mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{P}^N_k$ by the formula:

$$S([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [x_0y_0 : x_0y_1 : \ldots : x_ny_m]$$

(1) Show that S is well-defined and injective.
(2) Let $Z = V(T_{ij}T_{kl} - T_{il}T_{kj},\ 0 \leq i, k \leq n,\ 0 \leq j, l \leq m) \subseteq \mathbb{P}_k^N$. Show that $S(\mathbb{P}_k^n \times \mathbb{P}_k^m) = Z$ (more specifically, $S(U_i \times V_j) = Z \cap W_{ij}$ for all i, j).
(3) Show that the topology induced on $\mathbb{P}_k^n \times \mathbb{P}_k^m$ by the Zariski topology of $\mathbb{P}_k^N$ via the Segre embedding is different from the product topology.

Solution 6.

(1) To see that S is well-defined, note that

$$((\lambda x_0)y_0, (\lambda x_0)y_1, \dots, (\lambda x_n)y_m) = \lambda(x_0y_0, x_0y_1, \dots, x_ny_m)$$

and similarily if we replace $(y_0, \dots, y_m)$ by a scalar multiple. Hence the RHS in the definition of S doesn't depend on the choices of representatives, i.e. S is well-defined.

To see that S is injective, assume $S([x], [y]) = S([x'], [y'])$. Take i, j such that $x_i \neq 0$ and $y_j \neq 0$. Without loss of generality, we may assume $x_i = 1$ and $y_j = 1$. Then $x'_iy'_j = x_iy_j = 1 \neq 0$ and thus $x'_i \neq 0$. But then for all l, we have $y'_l = \lambda y_l$ so $y = y'$. Apply the same argument to y_j to get $x = x'$.

(2) If h_{ijkl} denotes the polynomial $T_{ij}T_{kl} - T_{il}T_{kj}$, we have

$$h_{ijkl}(x_0y_0, x_0y_1, \dots, x_ny_m) = (x_iy_j)(x_ky_l) - (x_iy_l)(x_ky_j) = 0.$$

Hence the image of S is contained in Z.

If $x_i \neq 0$, $y_j \neq 0$ then $S(x, y)_{ij} = x_iy_j \neq 0$. Hence $S(U_i \cap V_j) \subseteq W_{ij}$ and thus $S(U_i \cap V_j) \subseteq Z \cap W_{ij}$ by the above.

For the reverse inclusion, let $[z] = [z_{00}, z_{01}, \dots, z_{mn}] \in Z \cap W_{ij}$. As $z_{ij} \neq 0$, we may assume $z_{ij} = 1$. Set for all i',

$$x_{i'} \coloneqq z_{i'j}$$

and for all j',

$$y_{j'} \coloneqq z_{ij'}.$$

We then obtain

$$\underbrace{z_{ij}}_{=1} z_{i'j'} = z_{i'j}z_{ij'} = x_{i'}y_{j'}$$

for all i', j', i.e. $S([x], [y]) = [z]$. As $x_i = y_j = 1$ we have $[x] \in U_i$ and $[y] \in V_j$, so we conclude $Z \cap W_{ij} \subseteq S(U_i \times V_j)$.

In conclusion we have $Z \cap W_{ij} = S(U_i \times V_j)$ for all i, j, which in particular shows that $S(\mathbb{P}^n \times \mathbb{P}^m) = Z$.

(3) We can use the familiar example of the diagonal $\Delta \subseteq \mathbb{A}^n \times \mathbb{A}^n$, which isn't closed for the product topology (as $\mathbb{A}^n$ isn't Hausdorff by Exercise 3.3.2), but which is closed for the Zariski topology. The key in this is that $\mathbb{A}^n$ is irreducible, so any two non-empty open subsets intersect non-trivially. The same is true in $\mathbb{P}^n$, so the diagonal $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is not closed for the product topology.

Assume without loss of generality that $n \leq m$, and consider the subset

$$\Delta' \coloneqq \{([x_0 : \ldots : x_n], [x_0 : \ldots : x_n : 0 : \ldots : 0]) \mid [x_0 : \ldots : x_n] \in \mathbb{P}^n\} \subseteq \mathbb{P}^n \times \mathbb{P}^m.$$

There is a closed embedding $\mathbb{P}^n \to \mathbb{P}^m$ sending $[x_0 : \ldots : x_n]$ to $[x_0 : \ldots : x_n : 0 : \ldots : 0]$, which induces a closed embedding $i \colon \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^m$ for the product topology. As $i^{-1}(\Delta') = \Delta$ which isn't closed for the product topology, we conclude that Δ' isn't closed for the product topology either.

Nonetheless, let us show that $S(\Delta')$ is Zariski closed in $\mathbb{P}^N$. Indeed, it is a straightforward calculation to show that

$$S(\Delta') = Z \cap V(T_{ij} \mid n \leq j \leq m) \cap V(T_{ij} - T_{ji} \mid 0 \leq i, j \leq n).$$

In conclusion, $\Delta' \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed for the topology induced by the Zariski topology on $\mathbb{P}^N$ under S, but it isn't closed for the product topology.